Scalar-torsion Mode in Poincaré Gauge Theory

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Introduction

Geometric Meaning of Poincaré Gauge Theory

General Relativity ▶ (M, g, ∇), Gauge Theories ▶ (P, M, G, ω)

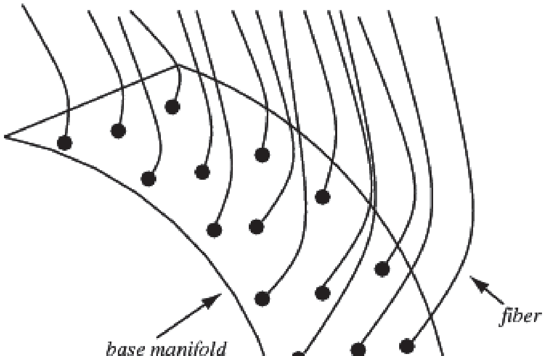
QED:$G = U(1)$, Yang-Mills field:$G = SU(2)$, etc,...

To incorporate **Poincaré group as structure group** $G = \mathbb{R}^{1,3} \rtimes SO(1,3)$, we need to regard

Definition 1 *(Frame Bundle)*

1. *Let* $L_x(M) = \{$*all frames on* $T_x M, x \in M\}$, $L(M) = \cup L_x(M)$. *Thus* $(L(M), M, G = GL(n, \mathbb{R}), \omega)$ *is a* ***principal bundle****.*

2. *In particular,* ***orthonormal frame (tetrads) bundle*** $(F(M), M, SO(1,3), \omega)$



Definition 2 *(affine space)*

*Let $\mathbb{A}$ be a set with a **vector space** V act as **space of translations** if V acts **freely** and **transitively** on A. i.e, for $p, q \in \mathbb{A}, v \in V$, that $v + p = p \Leftrightarrow v = 0$, $\exists\, v$ s.t $p + v = q$.*

Idea

$\mathbb{R}^n \xrightarrow{GL(n,\mathbb{R})} \mathbb{R}^n$	$\mathbb{A}^n \xrightarrow{GA(n,\mathbb{R})} \mathbb{A}^n$
and we know $GA(n, \mathbb{R}) = \mathbb{R}^n \rtimes GL(n, \mathbb{R})$	
$\mathbb{R}^n \xrightarrow{extend}$ affine space $\mathbb{A}^n$	$T_x M \xrightarrow{extend}$ affine **tangent** space $\mathbb{A}_x M$

Definition 3 *(Affine Frame Bundle)*

Let $A_x(M) = \{$all affine frames on $T_x M, x \in M\}$, $A(M) = \cup A_x(M)$. Thus $(A(M), M, \mathbb{R}^n \rtimes GL(n, \mathbb{R}), \omega)$.

*Now, **affine orthonormal frame (tetrads) bundle*** $(AF(M), M, \mathbb{R}^{1,3} \rtimes SO(1, 3), \tilde{\omega})$

where $\tilde{\omega} \in \Lambda^1(A(M), \mathbb{R}^4 \oplus \mathfrak{so}(1, 3))$ known as (gauge potential).

Gauge Transformations on Affine Frame Bundle

Definition 4
***Generalized* affine connection** $\tilde{\omega}$*: connections on* $A(M)$ *s.t*

$$\underbrace{\gamma^*\tilde{\omega}}_{\mathbb{R}^4\oplus\mathfrak{so}(1,3)} = \underbrace{\omega}_{\mathfrak{so}(1,3)} + \underbrace{\theta}_{\mathbb{R}^4} \quad \text{where } \theta \text{ arbitrary}$$

Affine connection $\tilde{\omega}$*: connections on* $A(M)$ *s.t.*

$$\underbrace{\gamma^*\tilde{\omega}}_{\mathbb{R}^4\oplus\mathfrak{so}(1,3)} = \underbrace{\omega}_{\mathfrak{so}(1,3)} + \underbrace{\varphi}_{\text{canonical form}} \quad \text{where } \gamma : GL(n,\mathbb{R}) \to GA(n,\mathbb{R})$$

Theorem (Kobayashi): 1-1 correspondence **(affine connection)** $\tilde{\omega} \overset{1-1}{\Longleftrightarrow} \omega$ **(linear connection)**

Definition 5 *(Gauge Transformation = fibre motion)*
$f : P \to P$ ***automorphism*** *of a principal bundle st* $f(pg) = f(p)g$ *for all* $p \in P, g \in G$ *and* $\pi(p) = \pi(f(p))$.

Theorem (A.Trautman, 1979): There exists **gauge transformation** $f : P \to P$ such that **(generalized affine connection)** $\Longrightarrow$ **(affine connection)**.

Scalar-torsion Mode in Poincaré Gauge Theory

Quadratic PGT Lagrangian

Like traditional gauge theory **quadratic terms** $(F \wedge \star F)$ are contained:
Hayashi and Shirafuji 1980 considered quadratic Lagrangians. Decompose (Under Lorentz Group):

$$R_{ijk}{}^{l} \longrightarrow \text{6 irreducible components} \qquad T_{ij}{}^{k} \longrightarrow \text{3 irreducible components} \tag{1}$$

Nester *et al* consider **spin** 0^{+} **mode**: $i, j, k, l = 0, 1, 2, 3$ (coordinate indices)

$$\boxed{L_G = \frac{a_0}{2} R + \frac{b}{24} R^2 + \frac{a_1}{8} \left(T_{ijk} T^{ijk} + 2 T_{ijk} T^{kji} - 4 T_k T^k \right)} \tag{2}$$

Positivity requirement: $a_1 > 0, \quad b > 0$, and **assume** $R \neq -\frac{6\mu}{b}$ for $\mu = a_0 + a_1 > 0$
Torsion $T_{ij}{}^{k}$ is torn into 3 pieces:

$$T_{ij}{}^{k} = \cancel{\frac{4}{3} Q_{[ij]k}} + \frac{2}{3} T_{[i} g_{j]k} + \cancel{\frac{1}{3} \epsilon_{ijkm} P^m} \qquad \text{where} \tag{3}$$

$$T_i := T_{ij}{}^{j}; \qquad P_i := \frac{1}{2} \epsilon_{ijkm} T^{jkm}; \quad Q_{ijk} = T_{i(jk)} - \frac{1}{3} T_i g_{jk} + \frac{1}{3} g_{i(j} T_{k)} \tag{4}$$

Simple spin-0^+ mode

Variation wrt **gauge potentials**: $(\delta\vartheta^\alpha, \delta\omega^{\alpha\beta})$ yields the EOM:

$$\frac{\partial L_G}{\partial \vartheta^\alpha} + D\left(\frac{\partial L_G}{\partial T^\alpha}\right) = -t_\alpha \qquad \text{(GFE)}$$
$$\frac{\partial L_G}{\partial \omega^{\alpha\beta}} + D\left(\frac{\partial L_G}{\partial \Omega^{\alpha\beta}}\right) = -s_{\alpha\beta} \qquad \text{(TFE)} \tag{5}$$

Assume **spin source current** ${S_{ij}}^k \equiv 0$,then **(TFE)** tells us:

$$\frac{\partial R}{\partial x^i} = -\frac{2}{3}\left(R + \frac{6\mu}{b}\right) T_i, \quad P_i = 0, \quad Q_{ijk} = 0 \tag{6}$$

In FRW cosmology ($k = 0$), (5),(6) tells us

$$\dot{H} = \frac{\mu}{6a_1} R + \frac{1}{6a_1}\mathcal{T} - 2H^2 \quad \textbf{(GFE)}$$
$$\dot{\Phi} = -\frac{a_0}{2a_1} + \frac{1}{2a_1}\mathcal{T} - 3H\Phi + \frac{1}{3}\Phi^2 \tag{7}$$
$$\dot{R} = -\frac{2}{3}\left(R + \frac{6\mu}{b}\right)\Phi \quad \textbf{(TFE)}$$

with (ordinary) **matter density** and **pressure**

$$\mathcal{T}_{tt} = \rho_M = \frac{b}{18}\left(R + \frac{6\mu}{b}\right)(3H - \Phi)^2 - \frac{b}{24}R^2 - 3a_1 H^2, \qquad (8)$$
$$\mathcal{T} = 3p_M - \rho_M = -\rho_m$$

and **(torsion) effective dark energy**

$$\rho_T = 3\mu H^2 - \frac{b}{18}\left(R + \frac{6\mu}{b}\right)(3H - \Phi)^2 + \frac{b}{24}R^2; \quad p_T = \frac{1}{3}\left(\mu(R - \bar{R}) + \rho_T\right) \qquad (9)$$

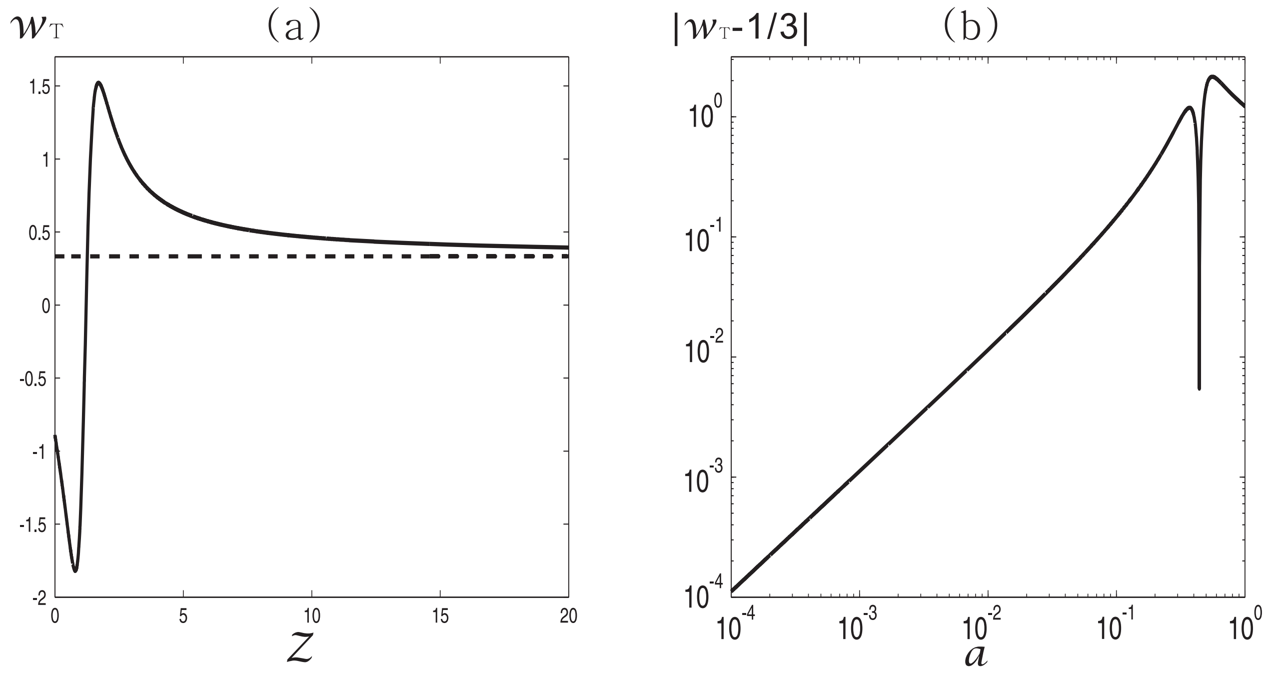
so that

$$H^2 = \frac{1}{3a_0}\left(\rho_M + \rho_T\right). \qquad (10)$$

To compute evolution $(H(t), \Phi(t), R(t), \rho_T(t))$, we need **numerical** method.

In particular, we concern **effective dark energy EoS** $\boxed{w_T = \frac{p_T}{\rho_T}}$

Asymptotic Behavior



(a) (b)

Figure 1: (a)Evolution w_T asymptotically to $1/3$ in high redshift, $a \ll 1$, with $(\tilde{a_0}, \tilde{a_1}, \tilde{\mu}, \tilde{H}_0, \tilde{R}_0, \chi) = (2, 1, 3, 2, 14, 3 \times 10^{-4})$

(b): $\ln|w_T - \frac{1}{3}| \simeq \ln\left(\frac{A_3}{3A_4}\right) + \ln a$

We can use **semi-analytical** method to explain. Let **Laurent expansion**

$$\rho_M = \frac{\rho_m^{(0)}}{a^3} + \frac{\rho_r^{(0)}}{a^4}; \qquad \frac{\rho_T}{\rho_m^{(0)}} = \sum_{k=-c}^{\infty} A_{-k}\, a^k \tag{11}$$

First we apply the following **rescaling**

$$\begin{aligned} &\tilde{a}_0 = a_0/m^2 b, \quad \tilde{a}_1 = a_1/m^2 b, \quad \tilde{t} = t \cdot m, \quad \tilde{\mu} = \tilde{a}_0 + \tilde{a}_1, \\ &\tilde{H}^2 = H^2/m^2, \quad \tilde{\Phi} = \Phi/m, \quad \tilde{R} = R/m^2, \end{aligned} \tag{12}$$

then **EOMs** read

$$\frac{d\tilde{H}}{d\tilde{t}} = \frac{\tilde{\mu}}{6\tilde{a}_1}\tilde{R} - \frac{\tilde{a}_0}{2\tilde{a}_1\, a^3} - 2\tilde{H}^2; \qquad \frac{d\tilde{\Phi}}{d\tilde{t}} = \frac{\tilde{a}_0}{2\tilde{a}_1}\left(\tilde{R} - \frac{3}{a^3}\right) - 3\tilde{H}\tilde{\Phi} + \frac{1}{3}\tilde{\Phi}^2, \tag{13}$$

$$\frac{d\tilde{R}}{d\tilde{t}} \simeq -\frac{2}{3}\tilde{R}\,\tilde{\Phi}; \qquad \frac{\tilde{R}}{18}\left(3\tilde{H} - \tilde{\Phi}\right) - \frac{\tilde{R}^2}{24} - 3\tilde{a}_1\tilde{H}^2 = 3\tilde{a}_0\left(\frac{1}{a^3} + \frac{\chi}{a^4}\right), \tag{14}$$

where $\chi = \rho_r^{(0)}/\rho_m^{(0)}$.

Semi-analytical Solution

Theorem 1 *In the high redshift regime* $(a \ll 1)$, $\rho_T = O(\frac{1}{a^4})$.

That is

$$\frac{\rho_T}{\rho_0^{(0)}} = \frac{A_4}{a^4} + \frac{A_3}{a^3} + \cdots + A_0 + A_{-1} a + \cdots$$

Using this theorem, we compute **EoS of effective (torsion) dark energy**

$$w_T = -1 - \frac{1}{3H}\frac{d}{dt}(\ln \rho_T) \overset{Thm\,1}{\simeq} -1 + \frac{1}{3}\left(\frac{4A_4 a^{-4} + 3A_3 a^{-3}}{A_4 a^{-4} + A_3 a^{-3}}\right) \simeq \frac{1}{3}\left(1 - \frac{A_3}{A_4} a\right) \qquad (15)$$

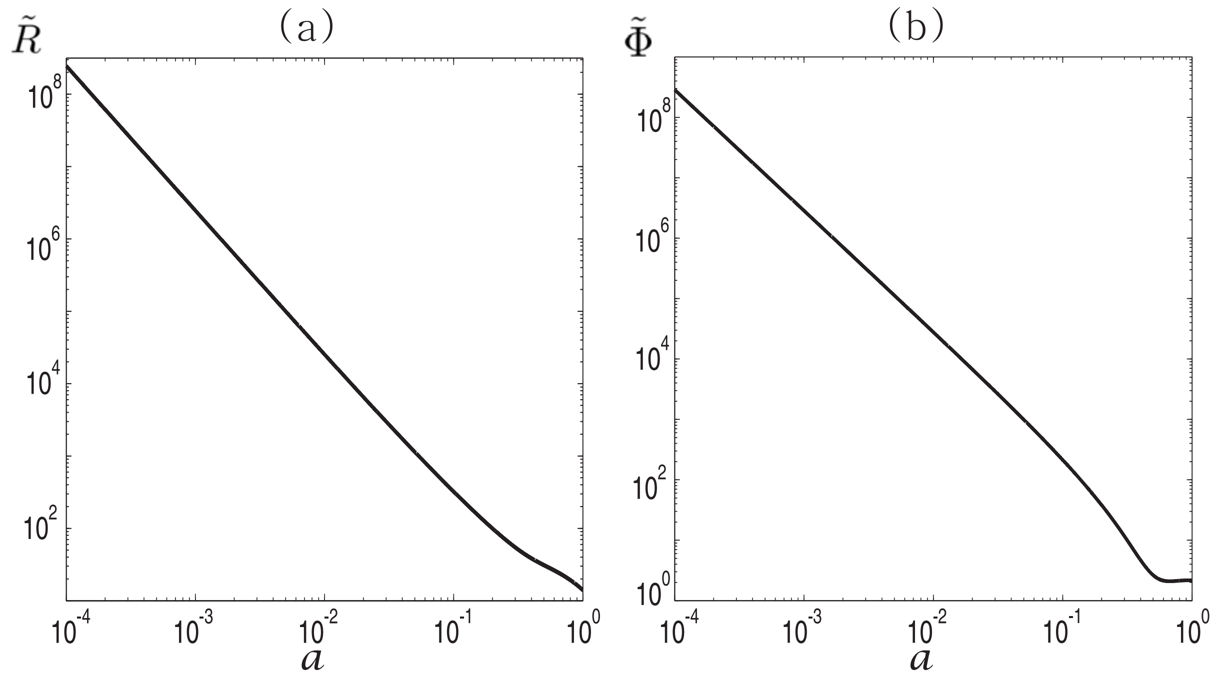
so that

$$w_T \longrightarrow \frac{1}{3} \quad \text{as} \quad a \to 0.$$

Comparison with Numerical Data

We can also calculate

$$\tilde{H}^2 \simeq (\chi + A_4)\, a^{-4}, \quad \tilde{R} \simeq \frac{2\tilde{a}_1}{\tilde{\mu}} A_2\, a^{-2}, \quad \tilde{\Phi} \simeq 3\tilde{H} \propto a^{-2}. \qquad (16)$$



(a) (b)

Figure 2: Both in $\log$ scale with $(\tilde{a_0}, \tilde{a_1}, \tilde{\mu}, \tilde{H}_0, \tilde{R}_0) = (2, 1, 3, 2, 14)$

Summary

We have seen

1. The geometric viewpoint of **Poincaré Gauge Theory in affine frame bundle**.
2. In the SNY-model, we use (semi-analytical) **Laurant expansion** to find an **asymptotic behavior in high redshift**. So that EoS $w_T \to \frac{1}{3}$ as $a \to 0$.
3. On the other hand, independent **numerical** computation **consents** the analytical result.

Theorem 2 *In the high redshift regime $(a \ll 1)$, $\rho_T = O(\frac{1}{a^4})$.*

sketch proof 1 *First we expand*

$$\tilde H^2(t) = \sum_{k=-c}^{\infty} r_k\, a^{k-4} \quad \textit{where } c \in \mathbb{Z} \tag{17}$$

so that we have

$$\frac{d\tilde H}{d\tilde t} = \sum_{k=-c}^{\infty} \left(\frac{k-4}{2}\right) r_k\, a^{k-4}. \tag{18}$$

(13) tells us

$$\tilde R = \frac{3\tilde a_1}{\tilde\mu}\left(\sum_{k=-c}^{\infty} k \cdot r_k\, a^{k-4}\right) + \frac{3\tilde a_0}{\tilde\mu\, a^3}, \tag{19}$$

$$\tilde\Phi = -\frac{3}{2}\tilde H \cdot \frac{\tilde a_1\left(\sum_{k=-c}^{\infty} k(k-4) r_k\, a^{k-4}\right) - \frac{3\tilde a_0}{a^3}}{\tilde a_1\left(\sum_{k=-c}^{\infty} k\, r_k\, a^{k-4}\right) + \frac{\tilde a_0}{a^3}}, \tag{20}$$

sketch proof 2

$$\left(\frac{\tilde{a}_1}{\tilde{\mu}}\right)^2 c^2 \cdot r_{-c}^3 \left[c^2 + (5 - \frac{\tilde{a}_0}{\tilde{\mu}})c + 4\right] = 0, \quad (21)$$

$\boxed{r_{-c} = 0 \text{ if } c \geq 1}$ *since* $0 < \tilde{a}_0/\tilde{\mu} < 1$ *and* $c^2 + (5 - \frac{\tilde{a}_0}{\tilde{\mu}})c + 4 \neq 0$, $\forall\, c \geq -1$. *Then*

$$\tilde{H}^2 = \frac{r_0}{a^4} + \frac{r_1}{a^3} + \frac{r_2}{a^2} + \frac{r_3}{a} + r_4 + \cdots \quad (22)$$

Finally, from (10)

$$\begin{aligned} \frac{\rho_T}{\rho_m^{(0)}} &= -\left(\frac{\chi}{a^4} + \frac{1}{a^4}\right) + \tilde{H}^2 \\ &= -\left(\frac{\chi}{a^4} + \frac{1}{a^3}\right) + \left(\frac{r_0}{a^4} + \frac{r_1}{a^3} + \frac{r_2}{a^2} + \frac{r_3}{a} + r_4 + \cdots\right) = O(\frac{1}{a^4}) \quad (r_0 \neq \chi) \end{aligned} \quad (23)$$